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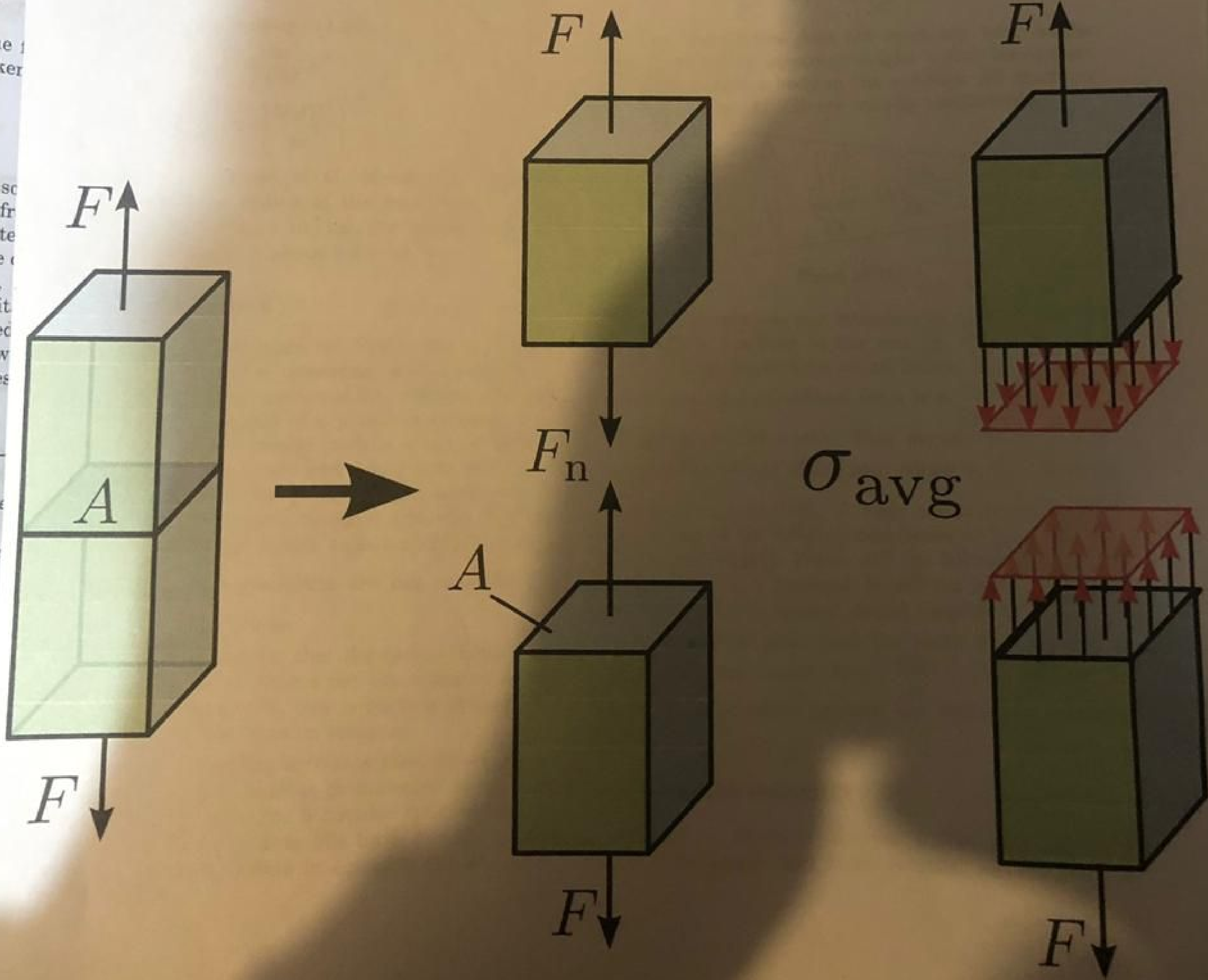
TOPIC No 1

Stress

Continuum Mechanics
M. Tech. structure
2nd semester

From. Prof. A. A. Masood
N. I. T. Sgr.

σ_{avg} = Average stress.



Chapter 1

Introduction

Continuum mechanics is a mathematical framework for studying the transmission of force through and deformation of materials of all types. The goal is to construct a framework that is free of special assumptions about the type of material, the size of deformations, the geometry of the problem and so forth. Of course, no real materials are actually continuous. We know from physics and chemistry that all materials are formed of discrete atoms and molecules. Even at much larger size scales, materials may be composed of distinct grains, e.g., a sand, or of grains of different constituents, e.g., steel, or deformable particles such as blood. Nevertheless, treating material as continuous is a great advantage since it allows us to use the mathematical tools of continuous functions, such as differentiation. In addtion to being convenient, this approach works remarkably well. This is true even at size scales for which the justfication of treating the material as a continuum might be debatable. The ultimate justification is that predictions made using continuum mechanics are in accord with observations and measurements.

Until recently, it was possible to solve a relatively small number of problems without the assumptions of small deformations and linear elastic behavior. Now, however, modern computational techniques have made it possible to solve problems involving large deformation and complex material behavior. This possibility has made it important to formulate these problems correctly and to be able to interpret the solutions. Continuum mechanics does this.

The vocabulary of continuum mechanics involves mathematical objects called tensors. These can be thought of as following naturally from vectors. Therefore, we will begin by studying vectors. Although most students are acquainted with vectors in some form or another, we will reintroduce them in a way that leads naturally to tensors.

Chapter 2

Vectors

Some physical quantities are described by scalars, e.g., density, temperature, kinetic energy. These are pure numbers, although they do have dimensions. It would make no physical sense to add a density, with dimensions of mass divided by length cubed, to kinetic energy, with dimensions of mass times length squared divided by time squared.

Vectors are mathematical objects that are associated with both a magnitude, described by a number, and a direction. An important property of vectors is that they can be used to represent physical entities such as force, momentum and displacement. Consequently, the meaning of the vector is (in a sense we will make precise) independent of how it is represented. For example, if someone punches you in the nose, this is a physical action that could be described by a force vector. The physical action and its result (a sore nose) are, of course, independent of the particular coordinate system we use to represent the force vector. Hence, the meaning of the vector is not tied to any particular coordinate system or description.

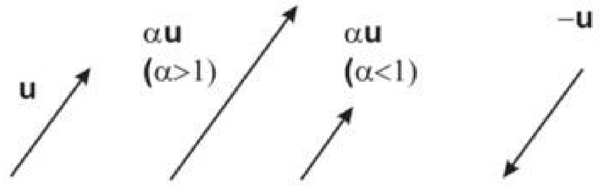
A vector $\mathbf{u}$ can be represented as a directed line segment, as shown in Figure 2.1. The length of the vector is denoted by u or by $|\mathbf{u}|$. Multiplying a vector by a positive scalar α changes the length or magnitude of the vector but not its orientation. If $\alpha > 1$, the vector $\alpha\mathbf{u}$ is longer than $\mathbf{u}$; if $\alpha < 1$, $\alpha\mathbf{u}$ is shorter than $\mathbf{u}$. If α is negative, the orientation of the vector is reversed. The addition of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \tag{2.1}$$

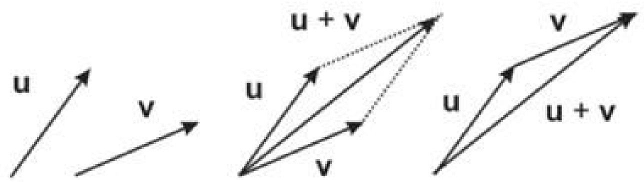
Although the same symbol is used as for ordinary addition, the meaning here is different. Vectors add according to the parallelogram law shown in Figure 2.1. It is clear from the construction that vector addition is commutative

$$\mathbf{w} = \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \tag{2.2}$$

Note the importance of distinguishing vectors from scalars; without the boldface denoting vectors, equation (2.1) would be incorrect: the magnitude of $\mathbf{w}$ is not



Multiplication of a vector by a scalar.



Addition of two vectors.

Figure 2.1: Multiplication of a vector by a scalar (top) and addition of two vectors (bottom).

the sum of the magnitudes of **u** and **v**. Alternatively the "tail" of one vector may be placed at the "head" of the other. The sum is then the vector directed from the free "tail" to the free "head". Implicit in both these operations is the idea that we are dealing with "free" vectors. In order to add two vectors, they can be moved, keeping the length and orientation, so that both vectors emanate from the same point or are connected head-to-tail.

The parallelogram rule for vector addition turns out to be a crucial property for vectors. Note that it follows from the nature of the physical quantities, e.g., velocity and force, that we represent by vectors. The rule for vector addition is also one way to distinguish vectors from other quantities that have both length and direction. For example, finite rotations about orthogonal axes can be characterized by length and magnitude but cannot be vectors because addition is not commutative (see Malvern, pp. 15-16). Hoffman (*About Vectors*, p. 11) relates the story of the tribe (now extinct) that thought spears were vectors because they had length and magnitude. To kill a deer to the northeast, they would throw two spears, one to the north and one to the east, depending on the resultant to strike the deer. Not surprisingly, there is no trace of this tribe, which only confirms the adage that "a little bit of knowledge can be a dangerous thing."

The procedure for vector subtraction follows from multiplication by a scalar and addition. To subtract **v** from **u**, first multiply **v** by -1, then add $-\mathbf{v}$ to **u**:

$$\mathbf{w} = \mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) \tag{2.3}$$

There are two ways to multiply vectors: the scalar or dot product and the vector or cross product. The scalar product is given by

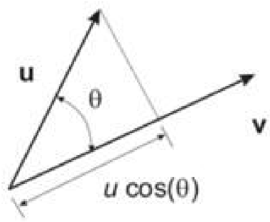
$$\mathbf{u} \cdot \mathbf{v} = uv\cos(\theta) \tag{2.4}$$

where θ is the angle between **u** and **v**. As indicated by the name, the result of this operation is a scalar. As shown in Figure 2.2, the scalar product is the magnitude of **v** multiplied by the projection **u** onto **v**, or vice versa. If $\theta = \pi/2$, the two vectors are *orthogonal*; if $\theta = \pi$, the two vectors are opposite in sense, i.e., their arrows point in opposite directions. The result of the vector or cross product is a vector

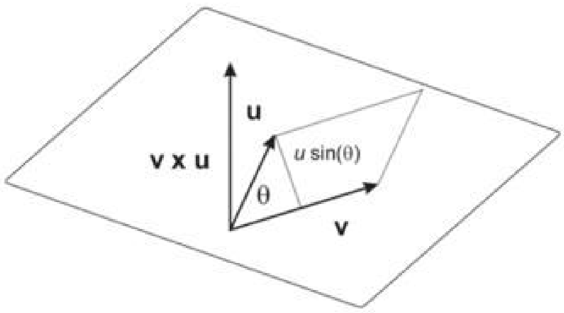
$$\mathbf{w} = \mathbf{u} \times \mathbf{v} \tag{2.5}$$

The magnitude of the result is $w = uv\sin(\theta)$, where θ is again the angle between **u** and **v**. As shown in Figure 2.2, the magnitude of the cross product is equal to the area of the parallelogram formed by **u** and **v**. The direction of **w** is perpendicular to the plane formed by **u** and **v** and the sense is given by the *right hand rule*: If the fingers of the right hand are in the direction of **u** and then curled in the direction of **v**, then the thumb of the right hand is in the direction of **w**. The three vectors **u**, **v** and **w** are said to form a right-handed system.

The triple vector product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ is equal to the volume of the parallelopiped formed by **u**, **v** and **w** if they are right-handed and minus the volume

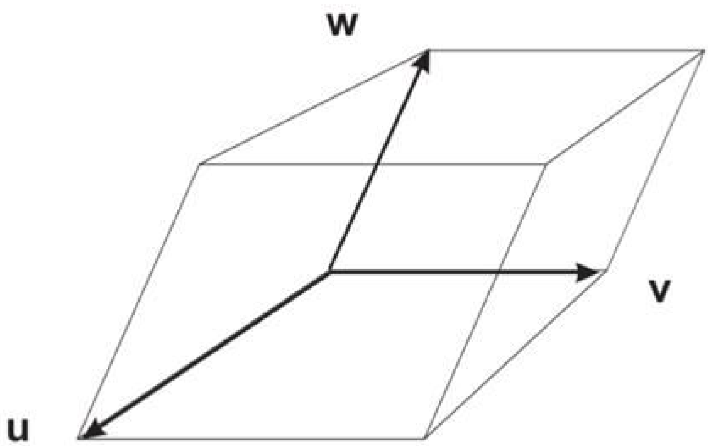


Scalar product of two vectors.



Cross product of two vectors.

Figure 2.2: Scalar and vector products.



The triple vector product **u x v • w** is the volume of the parallelpiped formed by the vectors if the order of the vectors is right-handed.

Figure 2.3: Triple vector product.

if they are not (Figure 2.3). The parenthesis in this expression can be omitted because it makes no sense if the dot product is taken first (because the result is a scalar and the cross product is an operation between two vectors).

Now consider the triple vector product $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$. The vector $\mathbf{v}\times\mathbf{w}$ must be perpendicular to the plane containing $\mathbf{v}$ and $\mathbf{w}$. Hence, the vector product of $\mathbf{v}\times\mathbf{w}$ with another vector $\mathbf{u}$ must result in a vector that is in the plane of $\mathbf{v}$ and $\mathbf{w}$. Consequently, the result of this operation can be can be represented as

$$\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \alpha\mathbf{v} + \beta\mathbf{w} \tag{2.6}$$

2.1 Additonal Reading

Chadwick, Chapter 1, Section 1; Malvern, Section 2.1, 2.2, 2.3; Reddy, 2.2.1 - 3.

Chapter 3

Tensors

A *tensor* is a linear, homogeneous vector-valued vector function. "Vector-valued vector function" means that a tensor operates on a vector and produces a vector as a result of the operation as depicted schematically in Figure 3.1. Hence, the action of a tensor $\mathbf{F}$ on a vector $\mathbf{u}$ results in another vector $\mathbf{v}$:

$$\mathbf{v} = \mathbf{F}(\mathbf{u}) \tag{3.1}$$

"Homogeneous" (of degree 1) means that the function $\mathbf{F}$ has the property

$$\mathbf{F}(\alpha\mathbf{u}) = \alpha F(\mathbf{u}) = \alpha\mathbf{v} \tag{3.2}$$

where α is a scalar. (Note: A function $f(x, y)$ is said to be homogeneous of degree n if $f(\alpha x, \alpha y) = \alpha^n f(x, y)$. A function $f(x, y)$ is linear if

$$f(x, y) = \alpha x + \beta y + c \tag{3.3}$$

Hence, $f(x, y) = \sqrt{x^2 + y^2}$ is homogeneous of degree one but not linear. Similarly, $f(x, y) = a(x + y) + c$ is linear but not homogeneous.) The function $\mathbf{F}$ is

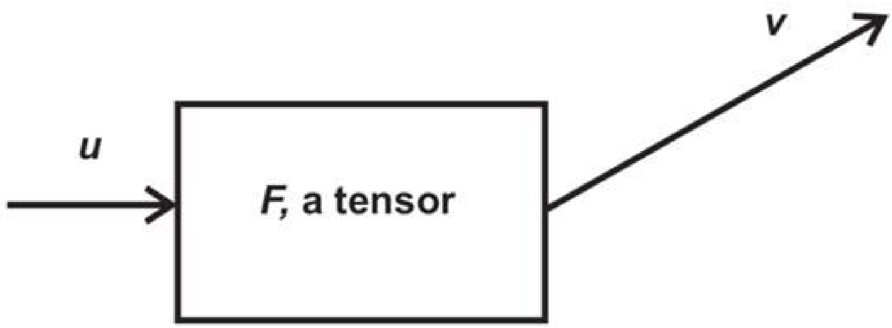


Figure 3.1: Schematic illustration of the effect of a tensor on a vector. The tensor acts on the vector $\mathbf{u}$ and outputs the vector $\mathbf{v}$.

"linear" if

$$\mathbf{F}(\mathbf{u}_1 + \mathbf{u}_2) = \mathbf{F}(\mathbf{u}_1) + \mathbf{F}(\mathbf{u}_2) = \mathbf{v}_1 + \mathbf{v}_2 \tag{3.4}$$

where $\mathbf{v}_1 = \mathbf{F}(\mathbf{u}_1)$ and $\mathbf{v}_2 = \mathbf{F}(\mathbf{u}_2)$

The definition of a tensor embodied by the properties (3.1), (3.2), and (3.4) suggests that a tensor can be represented in coordinate-free notation as

$$\mathbf{v} = \mathbf{F} \cdot \mathbf{u} \tag{3.5}$$

The operation denoted by the dot is defined by the properties (3.2), and (3.4). Therefore, if we want to determine if a "black box", a function $\mathbf{F}$, is a tensor, we input a vector $\mathbf{u}$ into the box. If the result of the operation represented by $\mathbf{F}$ is also a vector, say $\mathbf{v}$, then $\mathbf{F}$ must be a tensor. Since both sides of (3.5) are vectors, we can form the scalar product with another vector, say $\mathbf{w}$,

$$\mathbf{w} \cdot \mathbf{v} = \mathbf{w} \cdot \mathbf{F} \cdot \mathbf{u} \tag{3.6}$$

and the result must be a scalar. Because scalar multiplication of two vectors is commutative, the order of the vectors on the left side can be reversed. On the right side, it would be necessary to write $(\mathbf{F} \cdot \mathbf{u}) \cdot \mathbf{w}$. The parentheses indicate that the operation $\mathbf{F} \cdot \mathbf{u}$ must be done first; indeed, multiplying $\mathbf{u} \cdot \mathbf{w}$ first produces a scalar and the dot product of a scalar with a vector (or a tensor) is not an operation that is defined.

In contrast to the dot product of two vectors, the dot product of a tensor and a vector is not commutative. Reversing the order defines the *transpose* of the tensor $\mathbf{F}$ i.e.,

$$\mathbf{F} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{F}^T \tag{3.7}$$

Thus, it follows that

$$\mathbf{v} \cdot \mathbf{F} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{F}^T \cdot \mathbf{v} \tag{3.8}$$

where parentheses are not needed because the notation clearly indicates that the two vectors are not to be multiplied. If $\mathbf{F} = \mathbf{F}^T$, then the tensor $\mathbf{F}$ is said to be *symmetric*; if $\mathbf{F} = -\mathbf{F}^T$, then $\mathbf{F}$ is *antisymmetric* or *skew-symmetric*. Every tensor can be separated into the sum of a symmetric and a skew-symmetric tensor by adding and subtracting its transpose

$$\mathbf{F} = \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^T\right) + \frac{1}{2}\left(\mathbf{F} - \mathbf{F}^T\right) \tag{3.9}$$

Generally, the output vector $\mathbf{v}$ will have a different magnitude and direction from the input vector $\mathbf{u}$. In the special case that $\mathbf{v}$ is the same as $\mathbf{u}$, then for obvious reasons, the tensor is called the *identity* tensor and denoted $\mathbf{I}$. Hence, the identity tensor is defined by

$$\mathbf{u} = \mathbf{I} \cdot \mathbf{u} \tag{3.10}$$

for all vectors $\mathbf{u}$. Is it possible to operate our tensor black box in reverse? In terms of Figure 3.1, if we stick $\mathbf{v}$ in the right side, will we get $\mathbf{u}$ out the left?

The answer is "not always" although in many cases it will be possible for the particular tensors we are concerned with. Later we will determine the conditions for which the operation depicted in Figure 3.1 is reversible. If it is, then the operation defines the inverse of $\mathbf{F}$

$$\mathbf{u} = \mathbf{F}^{-1} \cdot \mathbf{v} \tag{3.11}$$

Substituting for $\mathbf{v}$ from (3.5) reveals that

$$\mathbf{F}^{-1} \cdot \mathbf{F} = \mathbf{I} \tag{3.12}$$

and that the dot product between two tensors produces another tensor.

If the output vector $\mathbf{v}$ has the same magnitude as the input vector $\mathbf{u}$, but a different direction, then the tensor operation results in a rotation

$$\mathbf{v} = \mathbf{R} \cdot \mathbf{u} \tag{3.13}$$

and the tensor is called *orthogonal* (for reasons we will see later). Because $\mathbf{u}$ and $\mathbf{v}$ have the same magnitudes

$$v^2 = \mathbf{v} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{u} = u^2$$

Using (3.7) to rewrite the left scalar product and (3.10) to rewrite the right gives

$$\mathbf{u} \cdot \mathbf{R}^T \cdot \mathbf{R} \cdot \mathbf{u} = \mathbf{u} \cdot \mathbf{I} \cdot \mathbf{u} \tag{3.14}$$

where no parentheses are necessary because the notation makes clear what is to be done. Because (3.14) applies for *any* vector $\mathbf{u}$, we can conclude that

$$\mathbf{R}^T \cdot \mathbf{R} = \mathbf{I} \tag{3.15}$$

and comparing with (3.12) reveals that the transpose of an orthogonal tensor is equal to its inverse. Physically, the rotation of a vector to another direction can always be reversed so we can expect the inverse to exist.

Is it possible to find an input vector $\mathbf{u}$ such that the output vector $\mathbf{v}$ has the same direction, but possibly a different magnitude? Intuitively, we expect that this is only possible for certain input vectors, if any. If the vector $\mathbf{v}$ is in the same direction as $\mathbf{u}$, then $\mathbf{v} = \lambda\mathbf{u}$, where λ is a scalar. Substituting in (3.5) yields

$$\mathbf{F} \cdot \mathbf{u} = \lambda\mathbf{u} \tag{3.16}$$

or

$$(\mathbf{F} - \lambda\mathbf{I}) \cdot \mathbf{u} = \mathbf{0} \tag{3.17}$$

If the inverse of $\mathbf{F} - \lambda\mathbf{I}$ exists then the only possible solution is $\mathbf{u} = \mathbf{0}$. Consequently there will be special values of λ and $\mathbf{u}$ that satisfy this equation only when the inverse does not exist. A value of λ that does so is an *eigenvalue* (*principal value, proper number*) of the tensor $\mathbf{F}$ and the corresponding direction given by $\mathbf{u}$ is the eigenvector (principal direction). It is clear from (3.17) that if $\mathbf{u}$ is a solution, then so is $\alpha\mathbf{u}$ where α is any scalar. Hence, only the direction of the eigenvector is determined. It is customary to acknowledge this by normalizing the eigenvector to unit magnitude, $\boldsymbol{\mu} = \mathbf{u}/u$.

Later we will learn how to determine the principal values and directions and their physical significance. But, because all of the tensors we will deal with are real and many of them are symmetric, we can prove that the eigenvalues and eigenvectors must have certain properties without having to determine them explicitly.

First we will prove that a real, symmetric 2nd order tensor has real eigenvalues. Let $\mathbf{T}$ be a real symmetric 2nd order tensor with eigenvalues λ_K, $K =$I, II, III and corresponding eigenvectors $\boldsymbol{\mu}_K$, $K =$ I, II, III. Then

$$\mathbf{T} \cdot \boldsymbol{\mu}_K = \lambda_K \, \boldsymbol{\mu}_K, \text{ (no sum on } K) \tag{3.18}$$

Taking complex conjugate of both sides gives

$$\bar{\mathbf{T}} \cdot \bar{\boldsymbol{\mu}}_K = \bar{\lambda}_K \bar{\boldsymbol{\mu}}_K, \text{ (no sum on } K) \tag{3.19}$$

Multipling (3.18) by $\bar{\boldsymbol{\mu}}_K$ yields

$$\bar{\boldsymbol{\mu}}_K \cdot \mathbf{T} \cdot \boldsymbol{\mu}_K = \lambda_K \bar{\boldsymbol{\mu}}_K \cdot \boldsymbol{\mu}_K, \text{ (no sum on } K) \tag{3.20}$$

and (3.19) by $\boldsymbol{\mu}_K$ yields

$$\boldsymbol{\mu}_K \cdot \bar{\mathbf{T}} \cdot \bar{\boldsymbol{\mu}}_K = \bar{\lambda}_K \bar{\boldsymbol{\mu}}_K \cdot \boldsymbol{\mu}_K, \text{ (no sum on } K) \tag{3.21}$$

Because $\mathbf{T} = \mathbf{T}^T$, the left hand sides are the same. Therefore, subtracting gives

$$0 = (\lambda_K - \bar{\lambda}_K) \bar{\boldsymbol{\mu}}_K \cdot \boldsymbol{\mu}_K, \text{ (no sum on } K) \tag{3.22}$$

Since $\bar{\boldsymbol{\mu}}_K \cdot \boldsymbol{\mu}_K \neq 0$, $\lambda_k = \bar{\lambda}_k$ and hence, the eigenvalues are real.

Now prove that the eigenvectors corresponding to distinct eigenvalues are orthogonal. For eigenvalue λ_I and corresponding eigenvector $\boldsymbol{\mu}_I$

$$\mathbf{T} \cdot \boldsymbol{\mu}_I = \lambda_I \, \boldsymbol{\mu}_I \tag{3.23}$$

and similarly for λ_{II} and $\boldsymbol{\mu}_{II}$

$$\mathbf{T} \cdot \boldsymbol{\mu}_{II} = \lambda_{II} \, \boldsymbol{\mu}_{II} \tag{3.24}$$

Dotting (3.23) with $\boldsymbol{\mu}_{II}$ and (3.24) with $\boldsymbol{\mu}_I$ yields

$$\boldsymbol{\mu}_{II} \cdot \mathbf{T} \cdot \boldsymbol{\mu}_I = \lambda_I \, \boldsymbol{\mu}_I \cdot \boldsymbol{\mu}_{II} \tag{3.25a}$$

$$\boldsymbol{\mu}_I \cdot \mathbf{T} \cdot \boldsymbol{\mu}_{II} = \lambda_{II} \, \boldsymbol{\mu}_{II} \cdot \boldsymbol{\mu}_I \tag{3.25b}$$

Because $\mathbf{T} = \mathbf{T}^T$ subtracting yields

$$(\lambda_I - \lambda_{II}) \boldsymbol{\mu}_I \cdot \boldsymbol{\mu}_{II} = 0 \tag{3.26}$$

Because the eigenvalues are assumed to be distinct $\lambda_I \neq \lambda_{II}$, and, consequently $\boldsymbol{\mu}_I \cdot \boldsymbol{\mu}_{II} = 0$. If $\lambda_I = \lambda_{II} \neq \lambda_{III}$, any vectors in the plane perpendicular to $\boldsymbol{\mu}_{III}$

can serve as eigenvectors. Therefore, it is always possible to find at least one set of orthogonal eigenvectors.

Lastly, we note the tensors we have introduced here are *second order tensors* because they input a vector and output a vector. We can, however, define nth order tensors $\mathbf{T}^{(n)}$ by the following recursive relation

$$\mathbf{T}^{(n)} \cdot \mathbf{u} = \mathbf{T}^{(n-1)} \tag{3.27}$$

If $\mathbf{T}^{(0)}$ is defined as a scalar then (3.27) shows that a vector can be considered as a tensor of order one. Later we will have occasion to deal with 3rd and 4th order tensors.

Chapter 4

Coordinate Systems

We have discussed a number of vector and tensor properties without referring at all to any particular coordinate system. Philosophically, this is attractive because it emphasizes the independence of the physical entity from a particular system. This process soon becomes cumbersome, however, and it is convenient to discuss vectors and tensors in terms of their components in a coordinate system. Moreover, when considering a particular problem or implementing the formulation in a computer, it is necessary to adopt a coordinate system.

Given that a coordinate system is necessary, we might take the approach that we should express our results on vectors in a form that is appropriate for any coordinate system. That is, we will make no assumptions that the axes of the system are orthogonal or scaled in the same way and so on. Indeed, this is often useful and can lead to a deeper understanding of vectors. Nevertheless, it requires the introduction of many details that, at least at this stage, will be distracting to our study of mechanics.

For the reasons just-discussed, we will consider almost exclusively rectangular cartesian coodinate systems. We will, however, continue to use and emphasize a coodinate free notation. Fortunately, results that can be expressed in a coordinate free notation, if interpreted properly, can be translated into any arbitrary coordinate system.

4.1 Base Vectors

A rectangular, cartesian coordinate system with origin O is shown in Figure 4.1. The axes are orthogonal and are labelled x, y, and z, or x_1, x_2 and x_3. A convenient way to specify the coordinate system is to introduce vectors that are tangent to the coordinate directions. More generally, a set of vectors is a *basis* for the space if every vector in the space can be expressed as a unique linear combination of the basis vectors. For rectangular cartesian systems, these base vectors can be chosen as unit vectors

$$|\mathbf{e}_1| = \mathbf{e}_1 \cdot \mathbf{e}_1 = 1, \ |\mathbf{e}_2| = |\mathbf{e}_3| = 1 \tag{4.1}$$

that are orthogonal:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \, \mathbf{e}_1 \cdot \mathbf{e}_3 = 0, \, \mathbf{e}_2 \cdot \mathbf{e}_3 = 0 \tag{4.2}$$

The six equations, (4.1) and (4.2), and the additional three that result from reversing the order of the dot product in (4.2) can be written more compactly as

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = \begin{cases} 1, & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} \tag{4.3}$$

where the indicies (i, j) stand for $(1, 2, 3)$ and δ_{ij} is the *Kronecker delta*. Therefore, (4.3) represents nine equations. Note that one i and one j appear on each side of the equation and that each index can take on the value 1, 2, or 3. Consequently, i and j in (4.3) are *free indicies*.

The projection of the vector $\mathbf{u}$ on a coordinate direction is given by

$$u_i = \mathbf{e}_i \cdot \mathbf{u} \tag{4.4}$$

where $i = 1, 2, 3$ and u_i is the scalar component of $\mathbf{u}$. We can now represent the vector $\mathbf{u}$ in terms of its components and the unit base vectors:

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 \tag{4.5}$$

Each term, e.g., $u_1\mathbf{e}_1$ is a vector component of $\mathbf{u}$. The left side of the equation is a *coordinate free* representation; that is, it makes no reference to a particular coordinate system that we are using to represent the vector. The right side is the component form; the presence of the base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ denote explicitly that u_1, u_2, and u_3 are the components with respect to the coordinate system with these particular base vectors. For a different coordinate system, with different base vectors, the right side would be different but would still represent the same vector, indicated by the coordinate free form on the left side.

4.2 Index Notation

The equation (4.5) can be expressed more concisely by using the summation sign:

$$\mathbf{u} = \sum_{k=1}^{3} u_k\mathbf{e}_k = u_k\mathbf{e}_k \tag{4.6}$$

where "k" is called a summation index because it takes on the explicit values 1, 2, and 3. It is also called a dummy index because it is simply a placeholder: changing "k" to "m" does not alter the meaning of the equation. Note that "k" appears twice on RHS but not on LHS. (In contrast, the free index "i" on the right side of (4.3) cannot be changed to "m" without making the same change

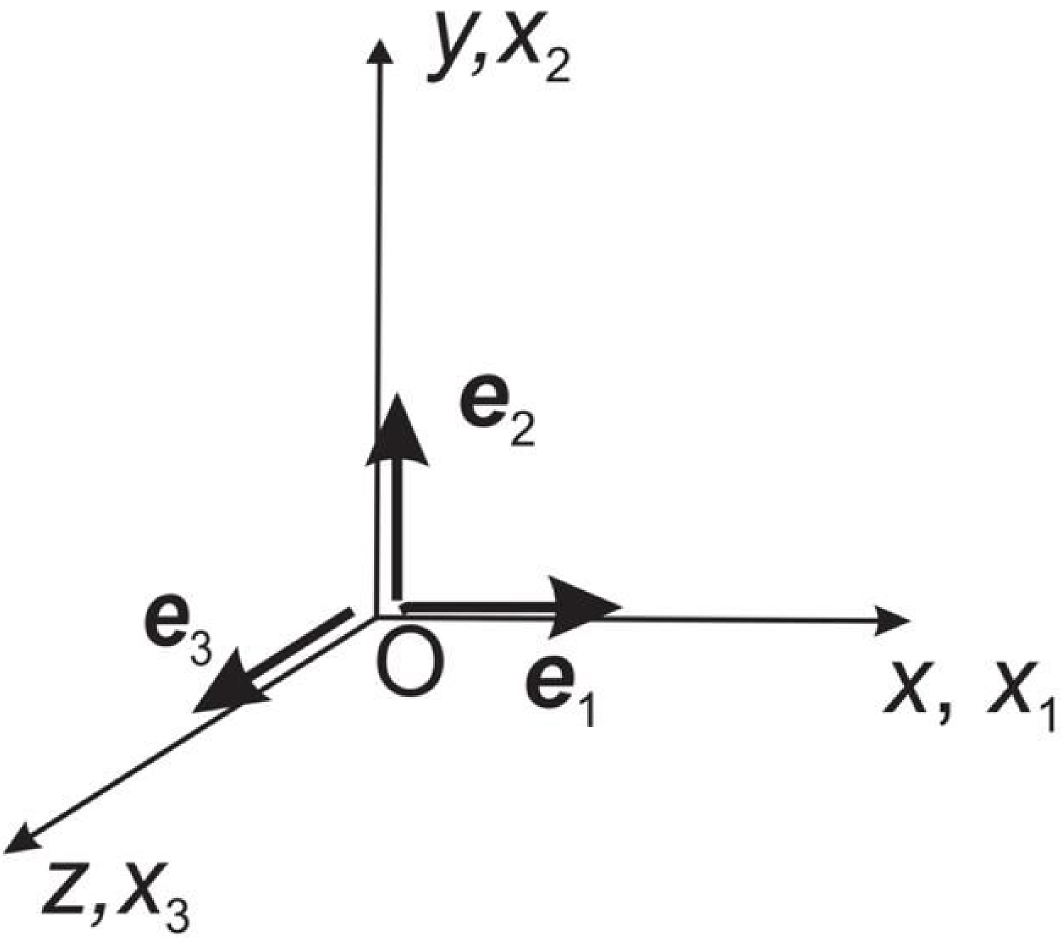


Figure 4.1: Rectangular, cartesian coordinate system specified by unit, orthogonal base vectors.

on the other side of the equation.) Because the form (4.6) occurs so frequently, we will adopt the *summation convention*: The summation symbol is dropped and summation is implied whenever an index is repeated in an additive term (a term separated by a plus or minus sign) on one side of the equation. This is a very compact and powerful notation but it requires adherence to certain rules. Regardless of the physical meaning of the equation, the following rules apply:

- A subscript should never appear more than twice (in an additive term) on one side of an equation.
- If a subscript appears once on one side of an equation it must appear exactly once (in each additive term) on the other side

For example, both of the following two equations are incorrect because the index "j" appears once on the right side but not at all on the left:

$$w_i = u_i + v_j \tag{4.7a}$$
$$w_i = u_k v_j s_k t_i \tag{4.7b}$$

The following equation is incorrect because the index "k" appears three times in an additive term:

$$w_{ij} = A_{ik} B_{jk} u_k \tag{4.8}$$

In contrast, the equation

$$a = u_k v_k + r_k s_k + p_k q_k \tag{4.9}$$

is correct. Even though "k" appears six times on the right side, it only appears twice in each additive term.

We can now use the scalar product, the base vectors and index notation to verify some relations we have obtained by other means. To determine the component of the vector $\mathbf{u}$ with respect to the ith coordinate direction we form the scalar product $\mathbf{e}_i \cdot \mathbf{u}$ and then express $\mathbf{u}$ in its component form:

$$\mathbf{e}_i \cdot \mathbf{u} = \mathbf{e}_i \cdot (u_j \mathbf{e}_j) \tag{4.10}$$

Note that it would be incorrect to write $u_i \mathbf{e}_i$ on the right side since the index i would then appear three times. The scalar product is an operation between vectors and, thus, applies to the two basis vectors. Their scalar product is given by (4.3). Recalling that the repeated j implies summation and using the property of the Kronecker delta (4.3) yields

$$\mathbf{e}_i \cdot \mathbf{u} = u_j (\mathbf{e}_i \cdot \mathbf{e}_j) \tag{4.11a}$$
$$= u_j \delta_{ij} = \sum_{j=1}^{3} \delta_{ij} u_j = \delta_{i1} u_1 + \delta_{i2} u_2 + \delta_{i3} u_3 \tag{4.11b}$$
$$= u_i \tag{4.11c}$$

Thus the inner product of a vector with a basis vector gives the component of the vector in that direction. This operation can be used to convert coordinate-free expressions to their cartesian component form. For example, the sum of two vectors is given by

$$\mathbf{w} = \mathbf{u} + \mathbf{v} \tag{4.12}$$

in the coordinate free notation. Dotting both sides with the base vectors $\mathbf{e}_i$ yields the component form

$$w_i = u_i + v_i \tag{4.13}$$

As a final example, consider the expression for the scalar product in terms of the components of the vectors:

$$\begin{aligned}
\mathbf{u} \cdot \mathbf{v} &= (u_i \mathbf{e}_i) \cdot (v_j \mathbf{e}_j) && (4.14a) \\
&= u_i v_j \, (\mathbf{e}_i \cdot \mathbf{e}_j) && (4.14b) \\
&= u_i v_j \delta_{ij} = \sum_{i=1}^{3} \sum_{j=1}^{3} u_i v_j \delta_{ij} = \sum u_j v_j = u_k v_k && (4.14c)
\end{aligned}$$

4.3 Tensor Components

The definition of a tensor embodied by the properties (3.1), (3.2), and (3.4) suggests that a tensor can be represented in coordinate-free notation as

$$\mathbf{v} = \mathbf{F} \cdot \mathbf{u} \tag{4.15}$$

The cartesian component representation follows from the procedure for identifying the cartesian components of vectors, i.e.,

$$\begin{aligned}
v_k &= \mathbf{e}_k \cdot \mathbf{v} = \mathbf{e}_k \cdot \{\mathbf{F} \cdot u_l \mathbf{e}_l\} && (4.16) \\
&= (\mathbf{e}_k \cdot \mathbf{F} \cdot \mathbf{e}_l) u_l
\end{aligned}$$

The second line can be represented in the component form

$$v_k = F_{kl} u_l \tag{4.17}$$

or in the matrix form

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{4.18}$$

where the

$$F_{kl} = \mathbf{e}_k \cdot \mathbf{F} \cdot \mathbf{e}_l \tag{4.19}$$

are the cartesian components of the tensor $\mathbf{F}$ (with respect to the base vectors $\mathbf{e}_l$).

4.4 Additional Reading

Chadwick, Chapter 1, Section 1; Malvern, Sections 2.1, 2.2, 2.3; Aris, 2. - 2.3. Reddy, 2.2.4 - 5.

Chapter 8

Change of Orthonormal Basis

Consider the two coordinate systems shown in Figure 8.1: the 123 system with base vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and the $1'2'3'$ system with base vectors $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$. In Chapter 3 we noted that an orthogonal tensor is one that rotates a vector without changing its magnitude. Thus we can use an orthogonal tensor to relate the base vectors in the two systems.

The base vectors in the primed and unprimed systems are related by

$$\mathbf{e}'_j = \mathbf{A} \cdot \mathbf{e}_j \tag{8.1}$$

where $\mathbf{A}$ is an orthogonal tensor. Forming the dot product in (8.1) gives

$$\mathbf{e}_i \cdot \mathbf{e}'_j = \cos(i, j') = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j = A_{ij} \tag{8.2}$$

where $\cos(i, j')$ is the cosine of the angle between the i axis and the j' axis. Thus, in the component A_{ij}, the second subscript (j in this case) is associated with the primed system. Either (8.1) or (8.2) leads to the dyadic representation

$$\mathbf{A} = \mathbf{e}'_k \mathbf{e}_k \tag{8.3}$$

Because both the new system and the old system of base vectors is orthonormal

$$\begin{aligned} \mathbf{e}'_i \cdot \mathbf{e}'_j &= \delta_{ij} = (\mathbf{A} \cdot \mathbf{e}_i) \cdot (\mathbf{A} \cdot \mathbf{e}_j) \\ &= \left(\mathbf{e}_i \cdot \mathbf{A}^T\right) \cdot (\mathbf{A} \cdot \mathbf{e}_j) \end{aligned}$$

the product

$$\mathbf{A}^T \cdot \mathbf{A} = \mathbf{I} \tag{8.4}$$

Thus, as also noted earlier, inverse of an orthogonal tensor is equal to its transpose. In index notation, (8.4) is expressed as

$$A_{ik} A_{jk} = A_{ki} A_{kj} = \delta_{ij} \tag{8.5}$$

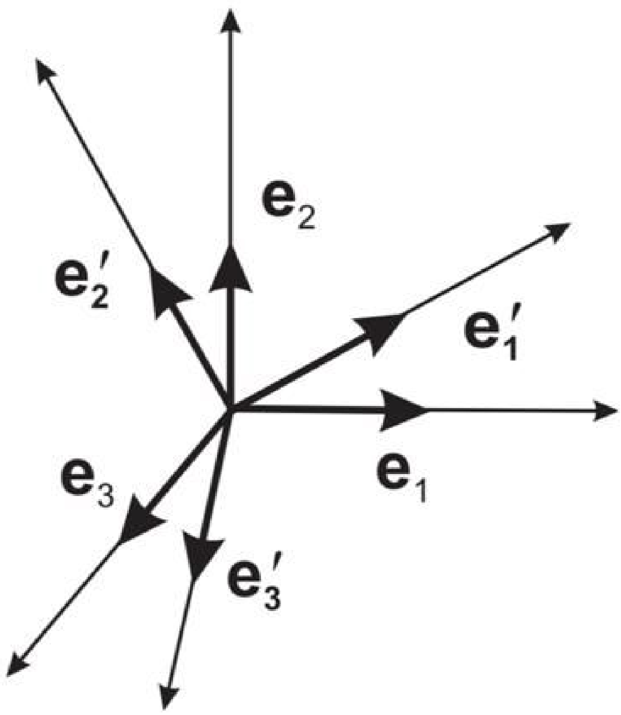


Figure 8.1: Rotation of the base vectors $\mathbf{e}_i$ to a new system $\mathbf{e}'_i$.

(where the first expression results from noting that the product in (8.4) can be taken in either order).

Consequently, the unprimed base vectors can be given in terms of the primed by

$$\mathbf{e}_m = \mathbf{A}^T \cdot \mathbf{e}'_m \tag{8.6}$$

and

$$\mathbf{e}_m \cdot \mathbf{e}'_n = \cos(m, n') = \mathbf{e}'_n \cdot \mathbf{A}^T \cdot \mathbf{e}'_m = A^T_{nm} = A_{mn} \tag{8.7}$$

which agrees with (8.2). These properties reinforce the choice of the name *orthogonal* for this type of tensor: it rotates one system of orthogonal unit vectors into another system of orthogonal unit vectors.

8.1 Change of Vector Components

Now consider a vector $\mathbf{v}$. We can express the vector in terms of components in either system

$$\mathbf{v} = v_i \mathbf{e}_i = v'_j \mathbf{e}'_j \tag{8.8}$$

since $\mathbf{v}$ represents the same physical entity. It is important to note that both the v_i and the v'_j represent the *same* vector; they simply furnish different descriptions. Given that the base vectors are related by (8.1) and (8.6), we wish to determine how the v_i and the v'_j are related. The component in the primed system is obtained by forming the scalar product of $\mathbf{v}$ with the base vector in the primed system:

$$\begin{aligned} v'_k &= \mathbf{e}'_k \cdot \mathbf{v} = \mathbf{e}'_k \cdot (v_i \mathbf{e}_i) && \text{(8.9a)} \\ &= v_i \mathbf{e}'_k \cdot \mathbf{e}_i && \text{(8.9b)} \\ &= v_i A_{ik} && \text{(8.9c)} \end{aligned}$$

The three equations (8.9) can also be represented as a matrix equation

$$\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \tag{8.10}$$

or, alternatively, as

$$\begin{bmatrix} v'_1 \\ v'_2 \\ v'_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{8.11}$$

Similarly, the components of $\mathbf{v}$ in the unprimed system can be expressed in terms of the components in the primed system

$$\begin{aligned} v_i &= \mathbf{e}_i \cdot \mathbf{v} = \mathbf{e}_i \cdot (v'_k \mathbf{e}'_k) && \text{(8.12a)} \\ &= (\mathbf{e}_i \cdot \mathbf{e}'_k)\, v'_k && \text{(8.12b)} \\ &= A_{ik} v'_k && \text{(8.12c)} \end{aligned}$$

or in matrix form

$$[v] = A\,[v'] \tag{8.13}$$

Note that the tensor $\mathbf{A}$ rotates the unprimed base vectors into the primed base vectors, according to (8.1), it is the components of $\mathbf{A}^T$ that appear in the matrix equation (8.11) as implied by the index form (8.9c). To interpret this result in another way rewrite (8.9a) as

$$\begin{aligned} v'_k &= \mathbf{e}'_k \cdot \mathbf{v} \\ &= (\mathbf{A} \cdot \mathbf{e}_k) \cdot \mathbf{v} \\ &= \left(\mathbf{e}_k \cdot \mathbf{A}^T\right) \cdot \mathbf{v} \\ &= \mathbf{e}_k \cdot \left(\mathbf{A}^T \cdot \mathbf{v}\right) \end{aligned}$$

Thus the v'_k are the components of the vector $\mathbf{A}^T \cdot \mathbf{v}$ on the *unprimed system*. This relation expresses the equivalence of rotating the coordinate system in one direction relative to a fixed vector and rotating a vector in the opposite direction relative to a fixed coordinate system.

8.2 Definition of a vector

Previously, we noted that vectors are directed line segments that add in a certain way. This property of addition reflects that nature of addition for the physical quanities that we represent as vectors, e.g. velocity and force. We now give another definition of a vector. This definition reflects the observation that the quantities represented by vectors are physical entities that cannot depend on the coordinate systems used to represent them. A (cartesian) vector $\mathbf{v}$ in three dimensions is a quantity with three components v_1, v_2, v_3 in the one rectangular cartesian system 0123, which, under rotation of the coordinates to another cartesian system $1'2'3'$ (Figure 8.1) become components v'_1, v'_2, v'_3 with

$$v'_i = A_{ji} v_j \tag{8.14}$$

where

$$A_{ji} = \cos(i', j) = \mathbf{e}'_i \cdot \mathbf{e}_j \tag{8.15}$$

This definition can then used to deduce other properties of vectors. For example, we can show that the sum of two vectors is indeed a vector. If $\mathbf{u}$ and $\mathbf{v}$ are vectors then $\mathbf{t} = \mathbf{u} + \mathbf{v}$ is a vector because it transforms like one:

$$t'_i = u'_i + v'_i = A_{ji} u_j + A_{ji} v_j \tag{8.16a}$$

$$= A_{ji}(u_j + v_j) = A_{ji} t_j \tag{8.16b}$$

8.3 Change of Tensor Components

Expressions for the components of $\mathbf{F}$ with respect to a different set of base vectors, say $\mathbf{e}'_k$, also follow from the relations for vector components:

$$v_k = F_{kl} u_l \tag{8.17}$$

and

$$\begin{aligned} v'_k &= A_{mk} F_{mn} u_n & (8.18a) \\ &= A_{mk} F_{mn} A_{nl} u'_l & (8.18b) \\ &= F'_{kl} u'_l & (8.18c) \end{aligned}$$

Because this result applies for *all* vectors **u** and **v**

$$F'_{kl} = A_{mk} F_{mn} A_{nl} \tag{8.19}$$

where, as before,

$$A_{mk} = \mathbf{e}_m \cdot \mathbf{e}_k^{'} = \cos(m, k^{'}) \tag{8.20}$$

This can be written in matrix form as

$$\left[F^{'}\right] = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} \begin{bmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{33} & A_{32} & A_{33} \end{bmatrix} \tag{8.21}$$

or

$$[F'] = [A]^T [F] [A] \tag{8.22}$$

Similarly, the inversion is given by

$$F_{ij} = A_{il} A_{jk} F'_{lk} \tag{8.23}$$

or

$$[F] = [A] [F'] [A]^T \tag{8.24}$$

The relations between components of a tensor in different orthogonal coordinate systems can be used as a second definition of a tensor that is analagous to the definition of a vector: In any rectangular coordinate system, a tensor is defined by nine components that transform according to the rule (8.19) when the relation between unit base vectors is (8.20).

As noted in Chapter 3, a symmetric tensor is one for which $\mathbf{T} = \mathbf{T}^T$. Because this relation can be expressed in coordinate-free form, we expect that the components are symmetric in any coordinate system. We can show this directly for rectangular cartesian systems using the relation (8.19). If the components of a tensor **T** are symmetric in one rectangular cartesian coordinate system, they are symmetric in any rectangular cartesian system:

$$\begin{aligned} \mathbf{T} &= T_{ij} \mathbf{e}_i \mathbf{e}_j \text{ where } T_{ij} = T_{ji} & (8.25) \\ T'_{kl} &= A_{ik} A_{jl} T_{ij} = A_{ik} A_{jl} T_{ji} & (8.26) \\ &= A_{jl} A_{ik} T_{ji} = A_{il} A_{jk} T_{ij} = T'_{lk} & (8.27) \end{aligned}$$

8.4 Additional Reading

Malvern, Sec. 2.4, Part 1, pp. 25-30; Chadwick, pp. 13 - 16; Aris 2.1.1, A.6. Reddy, 2.2.6.

Chapter 11

Traction and Stress Tensor

11.1 Types of Forces

We have already said that continuum mechanics assumes an actual body can be described by associating with it a mathematically continuous body. For example, we define the density at a point P as

$$\rho^{(P)} = \lim_{\Delta V \to 0} \frac{\Delta m}{\Delta V} \tag{11.1}$$

where ΔV contains the point P and Δm is the mass contained in ΔV. Continuum mechanics assumes that it makes sense or, at least is useful, to perform this limiting process even though we know that matter is discrete on an atomic scale. More precisely, ρ is the average density in a representative volume around the point P. What is meant by a representative volume depends on the material being considered. For example, we can model a polycrystalline material with a density that varies strongly from point-to-point in different grains. Alternatively, we might use a uniform density that reflects the density averaged over several grains.

Just as we have considered the mass to be distributed continuously, so also do we consider the forces to be continuously distributed. These may be of two types:

1. *Body forces* have a magnitude proportional to the mass, and act at a distance, e.g. gravity, magnetic forces (Figure 11.1). Body forces are computed per unit mass $\mathbf{b}$ or per unit volume $\rho\mathbf{b}$:

$$\mathbf{b}(\mathbf{x}) = \lim_{\Delta V \to 0} \frac{\mathbf{f}}{\rho \Delta V} \tag{11.2}$$

   The continuum hypothesis asserts that this limit exists, has a unique value, and is independent of the manner in which $\Delta V \to 0$.

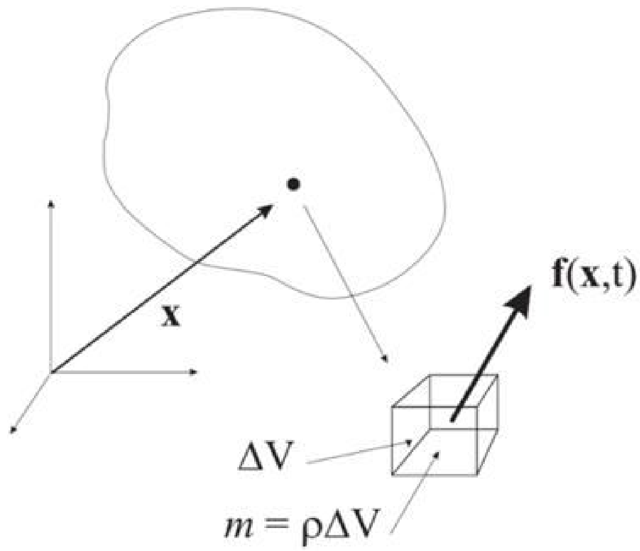


Figure 11.1: Illustration of the force $\mathbf{f}(\mathbf{x}, t)$ acting on the volume element ΔV.

2. Surface forces are computed per unit area and are contact forces. They may be forces that are applied to the exterior surface of the body or they may be forces transmitted from one part of a body to another.

Consider the forces acting on and within a body (Figure 11.2). Slice the body by a surface R (not necessarily planar) that passes through the point Q and divides the body into parts 1 and 2. Remove part 1 and replace it by the forces that 1 exerts on 2. The forces that 2 exerts on 1 are equal and opposite. Now consider the forces (exerted by 1 on 2) on a portion of the surface having area ΔS and normal $\mathbf{n}$ (at Q). From statics, we know that we can replace the distribution of forces on this surface by a statically equivalent force $\Delta\mathbf{f}$ and moment $\Delta\mathbf{m}$ at Q. Define the *average traction* on ΔS as

$$\Delta\mathbf{t}^{(\text{avg})} = \frac{\Delta\mathbf{f}}{\Delta S} \tag{11.3}$$

Now shrink C keeping point Q contained in C. Define traction at a point Q by

$$\mathbf{t}^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta\mathbf{f}}{\Delta S} \tag{11.4}$$

This is a vector (sometimes called "stress vector") and equals the force per unit area (intensity of force) exerted at Q by the material of 1 (side into which $\mathbf{n}$ points) on 2. In addition, we will assume that

$$\lim_{\Delta S \to 0} \frac{\Delta\mathbf{m}}{\Delta S} = 0 \tag{11.5}$$

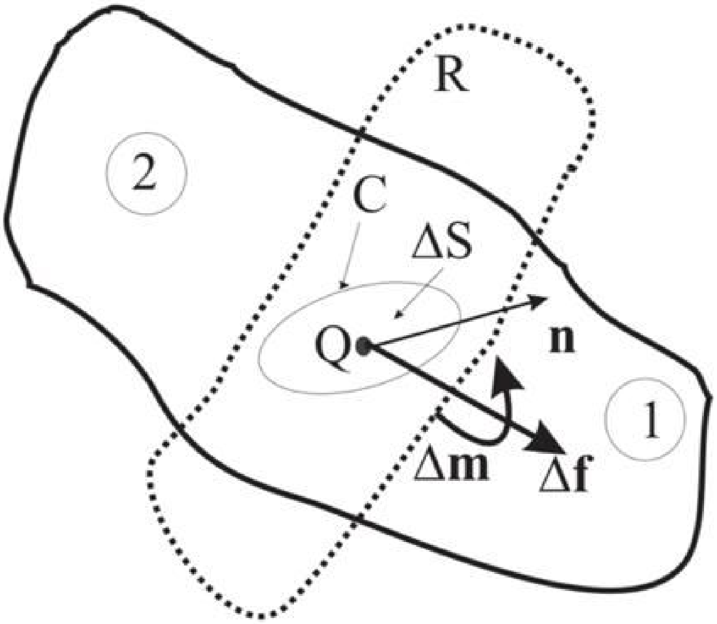


Figure 11.2: The surface R passes through the point Q and divides the body into two parts. The curve C contains Q and encloses an area ΔS. The unit normal to the surface at Q is $\mathbf{n}$. The net force exerted by 1 on 2 across ΔS is $\Delta \mathbf{f}$ and the net moment is $\Delta \mathbf{m}$.

This will necessarily be the case if the couple is due to distributed forces. The theory of couple stresses does not make this assumption.

In taking the limit (11.4), we have assumed the following:

1. Body is continuous.
2. $\Delta\mathbf{f}$ varies continuously.
3. No concentrated force at Q.
4. Limit is independent of the manner in which $\Delta S \to 0$ and the choice of the surface ΔS as long as the normal at Q is unique.

Note that traction is a vector and will have different values for different orientations of the normal $\mathbf{n}$ (through the same point) and different values at different points of the surface.

11.2 Traction on Different Surfaces

The traction at a point depends on the orientation of the normal. More specifically, the traction will be different for different orientations of the normal through the point. To investigate the dependence on the normal, we will use Newton's 2nd law

$$\sum \mathbf{F} = m\frac{d\mathbf{v}}{dt} \tag{11.6}$$

where $\mathbf{F}$ is the force, m is the mass, and $\mathbf{v}$ is the velocity. Now apply this to a slice of material of thickness h and area ΔS (Figure 11.3):

$$\mathbf{t}^{(\mathbf{n})}\Delta S + \mathbf{t}^{(-\mathbf{n})}\Delta S + \rho\mathbf{b}\Delta S h = \rho\Delta S h\frac{d\mathbf{v}}{dt} \tag{11.7}$$

where we have written the mass as $\rho\Delta S h$. Dividing through by ΔS yields

$$\mathbf{t}^{(\mathbf{n})} + \mathbf{t}^{(-\mathbf{n})} + \rho\mathbf{b}h = \rho h\frac{d\mathbf{v}}{dt} \tag{11.8}$$

Letting $h \to 0$ yields

$$\mathbf{t}^{(n)} = -\mathbf{t}^{(-n)} \tag{11.9}$$

Thus, the traction vectors are equal in magnitude and opposite in sign on two sides of a surface. In other words, reversing the direction of the normal to the surface reverses the sign of the traction vector. We can express the traction on planes normal to the coordinate directions $\mathbf{t}^{(\mathbf{e}_i)}$ in terms of their components

$$\mathbf{t}^{(\mathbf{e}_1)} = T_{11}\mathbf{e}_1 + T_{12}\mathbf{e}_2 + T_{13}\mathbf{e}_3 \tag{11.10a}$$
$$\mathbf{t}^{(\mathbf{e}_2)} = T_{21}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{23}\mathbf{e}_3 \tag{11.10b}$$
$$\mathbf{t}^{(\mathbf{e}_3)} = T_{31}\mathbf{e}_1 + T_{32}\mathbf{e}_2 + T_{33}\mathbf{e}_3 \tag{11.10c}$$

These three equations can be written as

$$\mathbf{t}^{(\mathbf{e}_i)} = T_{ij}\mathbf{e}_j \tag{11.11}$$

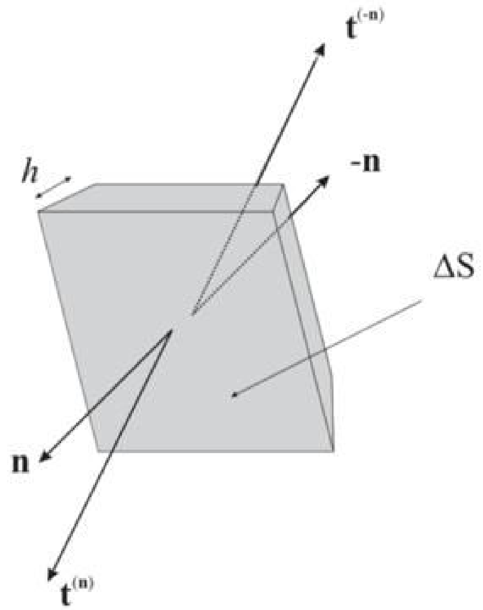


Figure 11.3: Tractions acting on opposite sides of a thin slice of material.

where the first index i denotes the direction of the normal to the plane on which the force acts and the 2nd index j denotes the direction of the force component. We can also express the traction as the scalar prodct of $\mathbf{e}_i$ with a tensor.

$$\mathbf{t}^{(i)} = \mathbf{e}_i \cdot (T_{mn}\mathbf{e}_m\mathbf{e}_n) \tag{11.12}$$

The term in parentheses is the stress tensor $\mathbf{T}$ and the T_{ij} are its cartesian components. T_{11}, T_{22}, T_{33} are *normal stresses*, and $T_{12}, T_{21}, T_{32}, T_{23}, T_{31}, T_{13}$ are *shear stresses*. Typically, in engineering, normal stresses are positive if they act in tension. In this case a stress component is positive if it acts in the positive coordinate direction on a face with outward normal in the positive coordinate direction or if it acts in the negative coordinate direction on a face with outward normal in the negative coordinate direction (Note that for a bar in equilibrium the forces acting on the ends of the bar are in opposite directions but these correspond to stress components of the same sign.).Often, in geology or geotechnical engineering, the sign convention is reversed because normal stresses are typically compressive.

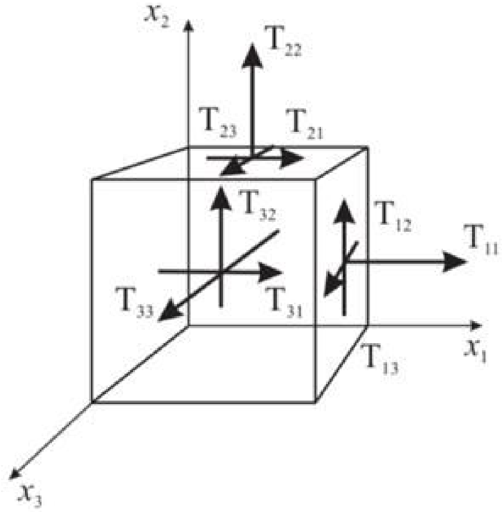


Figure 11.4: Illustrates the labelling of the components of the stress tensor. Remember that the cube shown represents a *point*.

11.3 Traction on an Arbitrary Plane (Cauchy tetrahedron)

Equation (11.10) gives the tractions on planes with normals in the coordinate directions but we would like to determine the traction on a plane with a normal in an arbitrary direction. Figure 11.5 shows a tetrahedron with three faces perpendicular to the coordinate axes and the fourth (oblique) face with a normal vector **n**. The oblique face has area ΔS and the area of the other faces can be expressed as

$$\Delta S_i = n_i \Delta S \tag{11.13}$$

The volume of the tetrahedron is given by

$$\Delta V = \frac{1}{3} h \Delta S \tag{11.14}$$

where h is the distance perpendicular to the oblique face through the origin. Appling Newton's 2nd Law to this tetrahedron gives

$$\mathbf{t}^{(\mathbf{n})}\Delta S + (-\mathbf{t}^{(i)}\Delta S_i) + \rho \mathbf{b} \Delta V = \rho \Delta V \frac{d\mathbf{v}}{dt} \tag{11.15}$$

In the second term, we have used (11.9) to express the sum of the forces acting on the planes perpendicular to the negative of the coordinate directions. Divide through by ΔS and let $h \to 0$. The result is

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(i)} n_i = n_1 \mathbf{t}^{(1)} + n_2 \mathbf{t}^{(2)} + n_3 \mathbf{t}^{(3)} \tag{11.16}$$

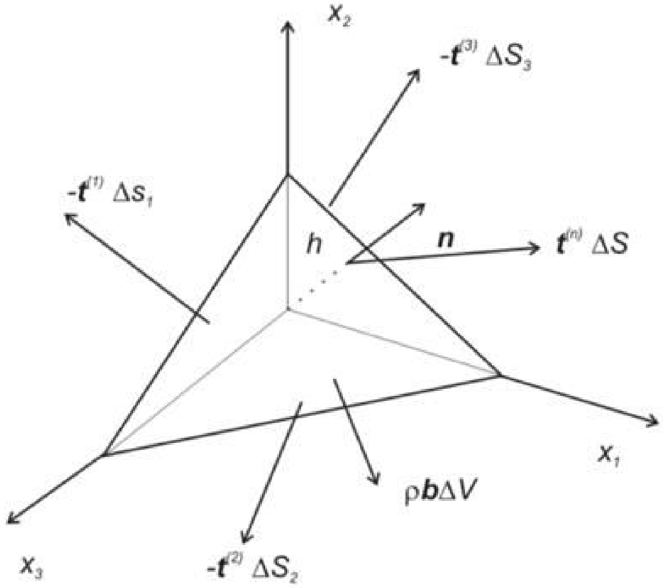


Figure 11.5: Tetrahedron with tractions acting on the faces.

Substituting (11.10) yields

$$\mathbf{t}^{(\mathbf{n})} = n_i T_{ij} \mathbf{e}_j = \mathbf{n} \cdot \mathbf{T} \tag{11.17}$$

This expression associates a vector $\mathbf{t}^{(\mathbf{n})}$ with every direction in space $\mathbf{n}$ by means of an expression that is linear and homogeneous and, hence, establishes $\mathbf{T}$ as a tensor. Since the $\mathbf{n}$ appears on the right side we will drop it as a superscript on $\mathbf{t}$ hereafter.

Because $\mathbf{T}$ is a tensor, its components in a rectangular cartesian system must transform accordingly

$$T'_{ij} = A_{pi} A_{qj} T_{pq} \tag{11.18}$$

where

$$A_{pi} = \mathbf{e}'_i \cdot \mathbf{e}_p \tag{11.19}$$

11.4 Symmetry of the stress tensor

We can also show that $\mathbf{T}$ is a symmetric tensor (later, we will give a more general proof) by enforcing that the sum of the moments is equal to the moment of inertia multiplied by the angular acceleration for a small cuboidal element centered at (x_1, x_2, x_3) with edges Δx_1, Δx_2 and Δx_3(not shown). For simplicity, consider the element to be subjected only to shear stresses T_{12} and T_{21} in the $x_1 x_2$ plane. The moment of inertia about the center of this element is

$$I = \frac{\rho}{12} \Delta x_1 \Delta x_2 \Delta x_3 (\Delta x_1^2 + \Delta x_2^2) \tag{11.20}$$

Summing the moments yields

$$\begin{aligned} &\left[T_{12}(x_1 + \frac{\Delta x_1}{2}, x_2)\Delta x_2 + T_{12}(x_1 - \frac{\Delta x_1}{2}, x_2)\Delta x_2\right] \Delta x_2 \Delta x_3 \frac{1}{2}\Delta x_1 \\ &- \left[T_{21}(x_1, x_2 + \Delta x_2 \frac{1}{2}) + T_{21}(x_1, x_2 - \frac{1}{2}\Delta x_2)\right] \Delta x_1 \Delta x_3 \frac{1}{2}\Delta x_2 \\ = &\ \alpha \frac{\rho}{12}(\Delta x_1 \Delta x_2 \Delta x_3)(\Delta x_1^2 + \Delta x_2^2) \end{aligned} \tag{11.21}$$

where the $\Delta x_1/2$ and $\Delta x_2/2$ in the first two lines are the moment arms. Dividing through by $\Delta x_1 \Delta x_2 \Delta x_3$ and letting $\Delta x_1 \Delta x_2 \to 0$ yields

$$T_{21} = T_{12} \tag{11.22}$$

and, similarly,

$$T_{ij} = T_{ji} \tag{11.23}$$

Later we will give a more general derivation of this result and see that it does not pertain when the stress is defined per unit reference (as distinguished from current) area.

11.5 Additional Reading

Malvern, 3.1, 3.2; Chadwick, 3.3; Aris, 5.11 - 5.15; Reddy, 4.1-4.2.

3. MEASURES OF STRESS

3.1 Mass and density

Mass is a physical variable associated with a body. At the intuitive level, mass is perceive be a measure of the amount of material contained in an arbitrary portion of body. As such *non-negative* scalar quantity independent of the time. Mass is *additive*, that is the mass of a b is the sum of the masses of its parts. These statetments imply the existence of a scalar fiel assigned to each particle $\mathcal{X}$ such that the mass of the body $\mathcal{B}$ currently occupying finite vol $v(\mathcal{B})$ is determined by

$$m(\mathcal{B}) = \int_{v(\mathcal{B})} \varrho dv \ . \qquad ($$

ϱ is called the *density* or the *mass density* of the material composing $\mathcal{B}$. As introduced, ϱ def the mass per unit volume. If the mass is not continuous in $\mathcal{B}$, then instead of (3.1) we write

$$m(\mathcal{B}) = \int_{v_1(\mathcal{B})} \varrho dv + \sum_{\alpha} m_{\alpha} \ , \qquad ($$

where the summation is taken over all *discrete* masses contained in the body. We shall be dea with a continuous mass medium in which (3.1) is valid, which implies that $m(\mathcal{B}) \to 0$ as $v(\mathcal{B})$ - We therefore have

$$0 \leq \varrho < \infty \ . \qquad ($$

3.2 Volume and surface forces

The forces that act on a continuum or between portions of it may be divided into *long-r* *forces* and *short-range forces.*

Long-range forces are comprised of *gravitational, electromagnetic* and *inertial* forces. T forces decrease very gradually with increasing distance between interacting particles. As a re long-range forces act uniformly on all matter contained within a sufficiently small volume, so t they are proportional to the volume size involved. In continuum mechanics, long-range forces referred to as *volume* or *body forces.*

The body force acting on $\mathcal{B}$ is specified by vector field $\vec{f}$ defined on the configuration $\mathcal{B}$. field is taken as measured per unit mass and is assumed to be continuous. The total body f acting on the body $\mathcal{B}$ currently occupying finite volume $v(\mathcal{B})$ is expressed as

$$\vec{F}(\mathcal{B}) = \int_{v(\mathcal{B})} \varrho \vec{f} dv \ . \qquad ($$

Short-range forces comprise several types of *molecular* forces. Their characteristic featu that they decrease extremely abruptly with increasing distance between the interacting parti Hence, they are of consequence only when this distance does not exceed molecular dimensions a result, if matter inside a volume is acted upon by short-range forces originating from interact with matter outside this volume, these forces only act upon a thin layer immediately belov surface. In continuum mechanics, short-range forces are called *surface* or *contact* forces and specified more closely by constitutive equations (Chapter 5).

3.3 Cauchy traction principle

A mathematical description of surface forces stems from the following Cauchy traction princ

We consider a material body $b(t)$ which is subject to body forces $\vec{f}$ and surface forces $\vec{g}$. p be an interior point of $b(t)$ and imagine a plane surface a^* passing through point p (sc times referred to as a *cutting plane*) so as to partition the body into two portions, design: I and II (Figure 3.1). Point p is lying in the area element Δa^* of the cutting plane, whic defined by the unit normal $\vec{n}$ pointing in the direction from Portion I into Portion II, as sh in Figure 3.1. The internal forces being transmitted across the cutting plane due to the ac of Portion II upon Portion I will give rise to a force distribution on Δa^* equivalent to a re tant surface force $\Delta\vec{g}$, as also shown in Figure 3.1. (For simplicity, body forces and surface fo acting on the body as a whole are not draw Figure 3.1.) Notice that $\Delta\vec{g}$ are not neces ily in the direction of the unit normal vecto The *Cauchy traction principle* postulates that limit when the area Δa^* shrinks to zero, wi remaining an interior point, exists and is give:

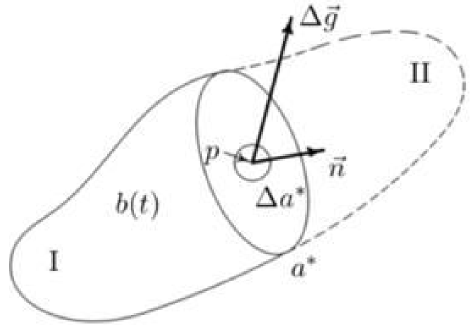


Figure 3.1.
Surface force on surface element Δa^*.

$$\vec{t}_{(\vec{n})} = \lim_{\Delta a^* \to 0} \frac{\Delta\vec{g}}{\Delta a^*} \; . \qquad ($$

Obviously, this limit is meaningful only if degenerates not into a curve but into a poir The vector $\vec{t}_{(\vec{n})}$ is called the *Cauchy stress tor* or the *Cauchy traction vector* (force per area). It is important to note that, in general, $\vec{t}_{(\vec{n})}$ depends not only on the position of p on but also the orientation of surface Δa^*, i.e., on its external normal $\vec{n}$. This dependence is th fore indicated by the subscript $\vec{n}$. [9] Thus, for the infinity of cutting planes imaginable thro point p, each identified by a specific $\vec{n}$, there is also an infinity of associated stress vectors $\vec{t}_{(\vec{n}}$ a given loading of the body.

We incidentally mention that a continuous distribution of surface forces acting across s surface is, in general, equivalent to a resultant force and a resultant torque. In (3.5) we have m the assumption that, in the limit at p, the torque per unit area vanishes and therefore there i remaining concentrated torque, or *couple stress*. This material is called the *non-polar continu* For a discussion of couple stresses and polar media, the reader is referred to Eringen, 1967.

3.4 Cauchy lemma

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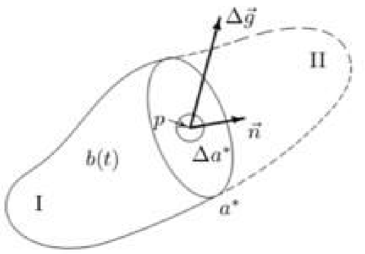

Figure 3.1.
Surface force on surface element Δa^*.

$$\vec{t}_{(\vec{n})} = \lim_{\Delta a^* \to 0} \frac{\Delta\vec{g}}{\Delta a^*} \,. \tag{3.5}$$

Obviously, this limit is meaningful only if Δa^* degenerates not into a curve but into a point p. The vector $\vec{t}_{(\vec{n})}$ is called the *Cauchy stress vector* or the *Cauchy traction vector* (force per unit area). It is important to note that, in general, $\vec{t}_{(\vec{n})}$ depends not only on the position of p on Δa^* but also the orientation of surface Δa^*, i.e., on its external normal $\vec{n}$. This dependence is therefore indicated by the subscript $\vec{n}$. [9] Thus, for the infinity of cutting planes imaginable through point p, each identified by a specific $\vec{n}$, there is also an infinity of associated stress vectors $\vec{t}_{(\vec{n})}$ for a given loading of the body.

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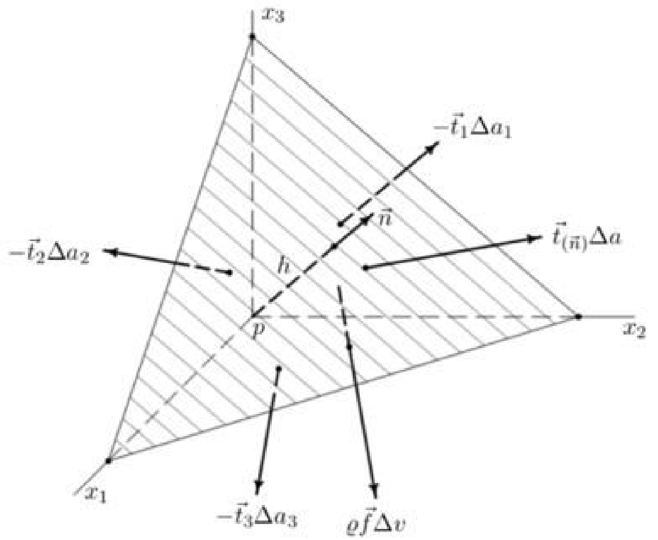

Figure 3.2. Equilibrium of an infinitesimal tetrahedron.

We now apply the equation of balance of linear momentum (Sect.4.1) to this tetrahedron,

$$\int_{\Delta v} \varrho\vec{f}dv - \int_{\Delta a_k} \vec{t}_k da_k + \int_{\Delta a} \vec{t}_{(\vec{n})} da = \frac{D}{Dt}\int_{\Delta v} \varrho\vec{v}dv \ .$$

The surface and volume integrals may be evaluated by the mean value theorem:

$$\varrho^*\vec{f}^*\Delta v - \vec{t}_k^*\Delta a_k + \vec{t}^*_{(\vec{n})}\Delta a = \frac{D}{Dt}\left(\varrho^*\vec{v}^*\Delta v\right) \ , \tag{3.6}$$

where ϱ^*, $\vec{f}^*$, and $\vec{v}^*$ are, respectively, the values of ϱ, $\vec{f}$, and $\vec{v}$ at some interior points of the tetrahedron and $\vec{t}^*_{(\vec{n})}$ and $\vec{t}^*_k$ are the values of $\vec{t}_{(\vec{n})}$ and $\vec{t}_k$ on the surface Δa and on coordinate surfaces Δa_k. The volume of the tetrahedron is given by

$$\Delta v = \frac{1}{3}h\Delta a \ , \tag{3.7}$$

where h is the perpendicular distance from point p to the base Δa. Moreover, the area vector $\Delta\vec{a}$ is equal to the sum of coordinate area vectors, that is,

$$\Delta\vec{a} = \vec{n}\Delta a = \Delta a_k\vec{i}_k \ . \tag{3.8}$$

Thus

$$\Delta a_k = n_k\Delta a \ . \tag{3.9}$$

Inserting (3.7) and (3.9) into (3.6) and canceling the common factor Δa, we obtain

$$\frac{1}{3}\varrho^*\vec{f}^*h - \vec{t}^*_k n_k + \vec{t}^*_{(\vec{n})} = \frac{1}{3}\frac{D}{Dt}\left(\varrho^*\vec{v}^*h\right) \ . \tag{3.10}$$

Now, letting the tetrahedron shrink to point p by taking the limit $h \to 0$ and noting that in this limiting process the starred quantities take on the actual values of those same quantities at point p, we have

$$\vec{t}_{(\vec{n})} = \vec{t}_k n_k \,, \tag{3.11}$$

which is the *Cauchy stress formula*. Equation (3.11) allows the determination of the Cauchy stress vector at some point acting across an arbitrarily inclined plane, if the Cauchy stress vectors acting across the three coordinate surfaces through that point are known.

The stress vectors $\vec{t}_k$ are, by definition, independent of $\vec{n}$. From (3.11) it therefore follows that

$$-\vec{t}_{(-\vec{n})} = \vec{t}_{(\vec{n})} \,. \tag{3.12}$$

The stress vector acting on a surface with the unit normal $\vec{n}$ is equal to the negative stress vector acting on the corresponding surface with the unit normal $-\vec{n}$. In Newtonian mechanics this statement is known as Newton's third law. The calculations show that this statement is valid for stress vector.

We now introduce the definition of the Cauchy stress tensor. The t_{kl} component of the *Cauchy stress tensor* $\boldsymbol{t}$ is given by the lth component of the stress vector $\vec{t}_k$ acting on the positive side of the kth coordinate surface:

$$\vec{t}_k = t_{kl}\vec{i}_l \qquad \text{or} \qquad t_{kl} = \vec{t}_k \cdot \vec{i}_l \,. \tag{3.13}$$

The first subscript in t_{kl} indicates the coordinate surface $x_k =$ const. on which the stress vector $\vec{t}_k$ acts, while the second subscript indicates the direction of the component of $\vec{t}_k$. For example, t_{23} is the x_3-components of the stress vector $\vec{t}_2$ acting on the coordinate surface $x_2 =$ const.. Now, if the exterior normal of $x_2 =$ const. points in the positive direction of the x_2-axis, t_{23} points in the positive direction of the x_3-axis. If the exterior normal of $x_2 =$ const. is in the negative direction of the x_2-axis, t_{23} is directed in the negative direction of the x_3-axis. The positive stress components on the faces of a parallelepiped built on the coordinate surfaces are shown in Figure 3.3. The nine components t_{kl} of the Cauchy stress tensor $\boldsymbol{t}$ may be arranged in a matrix form

$$\boldsymbol{t} = \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} (\vec{i}_k \otimes \vec{i}_l) \,. \tag{3.14}$$

Considering (3.13), the Cauchy stress formula (3.11) reads

$$\vec{t}_{(\vec{n})} = \vec{n} \cdot \boldsymbol{t} \,, \tag{3.15}$$

which says that the Cauchy stress vector acting on any plane through a point is fully characterized as a linear function of the stress tensor at that point. The normal component of stress vector,

$$t_n = \vec{t}_{(\vec{n})} \cdot \vec{n} = \vec{n} \cdot \boldsymbol{t} \cdot \vec{n} \,, \tag{3.16}$$

is called the *normal stress* and is said to be *tensile* when positive and *compressive* when negative. The stress vector directed tangentially to surface has the form

$$\vec{t}_t = \vec{t}_{(\vec{n})} - t_n\vec{n} = \vec{n} \cdot \boldsymbol{t} - (\vec{n} \cdot \boldsymbol{t} \cdot \vec{n})\vec{n} \,. \tag{3.17}$$

The size of $\vec{t}_t$ is known as the *shear stress*. For example, the components t_{11}, t_{22} and t_{23} in Figure 3.3 are the normal stresses and the mixed components t_{12}, t_{13}, etc. are the shear stresses.

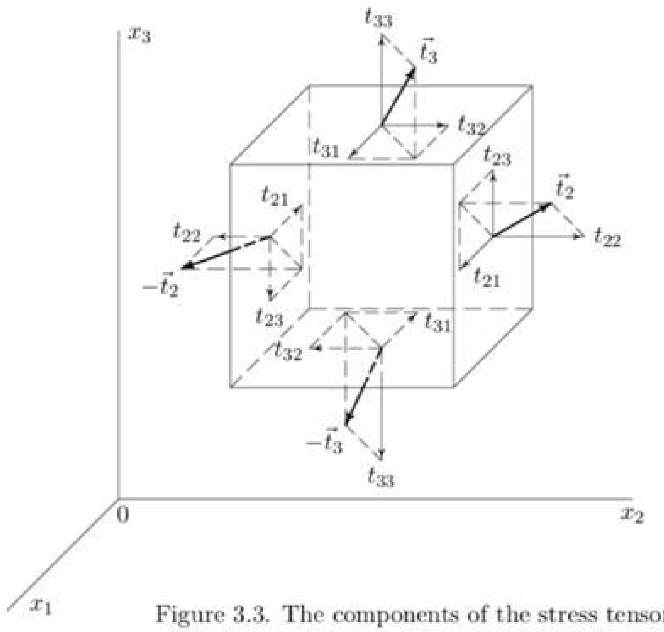

Figure 3.3. The components of the stress tensor.

If, in some configuration, the shear stress is identically zero and the normal stress is independent of $\vec{n}$, the stress is said to be *spherical*. In this case, there is a scalar field p, called the *pressure*, such that

$$\vec{t}_{(\vec{n})} = -p\vec{n} \qquad \text{and} \qquad \boldsymbol{t} = -p\boldsymbol{I}\ . \tag{3.18}$$

3.5 Other measures of stress

So far, we have represented short-range intermolecular forces in terms of the Cauchy stress vector $\vec{t}_k$ or tensor $\boldsymbol{t}$. There are, however, two other ways of measuring or representing these forces, each of which plays a certain role in the theory of continuum mechanics. The Eulerian Cauchy stress tensor gives the surface force acting on the deformed elementary area da in the form

$$d\vec{g} = \vec{t}_{(\vec{n})}da = (\vec{n}\cdot\boldsymbol{t})da = d\vec{a}\cdot\boldsymbol{t}\ . \tag{3.19}$$

The Cauchy stress tensor, like any other variable, has both an Eulerian and a Lagrangian description; the corresponding Lagrangian Cauchy stress tensor is defined by $\boldsymbol{T}(\vec{X},t) := \boldsymbol{t}(\vec{x}(\vec{X},t),t)$. We make, however, an exception in the notation and use $\boldsymbol{t}(\vec{X},t)$ for the Lagrangian description of the Cauchy stress tensor. The Eulerian Cauchy stress tensor $\boldsymbol{t}(\vec{x},t)$ arises naturally in the Eulerian form of the balance of linear momentum; the corresponding Lagrangian form of this principle cannot, however, be readily expressed in terms of the Lagrangian Cauchy stress tensor $\boldsymbol{t}(\vec{X},t)$.

A simple Lagrangian form of the balance of linear momentum can be obtained if a stress measure is referred to a surface in the reference configuration. This can be achieved by introducing the so-called *first Piola-Kirchhoff stress tensor* $\boldsymbol{T}^{(1)}$ as a stress measure referred to the referential area element $d\vec{A}$:

$$d\vec{g} = d\vec{a} \cdot \boldsymbol{t} =: d\vec{A} \cdot \boldsymbol{T}^{(1)} \,. \tag{3.20}$$

Here, the tensor $\boldsymbol{T}^{(1)}$ gives the surface force acting on the deformed area $d\vec{a}$ at $\vec{x}$ in terms of the corresponding referential element $d\vec{A}$ at the point $\vec{X}$. Thus, $\boldsymbol{T}^{(1)}$ is a measure of the force per unit referential area, whereas both the Eulerian and Lagrangian Cauchy stresses $\boldsymbol{t}(\vec{x},t)$ and $\boldsymbol{t}(\vec{X},t)$ are measures of the force per unit spatial area. The relationship between $\boldsymbol{T}^{(1)}$ and $\boldsymbol{t}$ is found using the transformation (1.75) between the deformed and undeformed elementary areas. The result can be expressed in either of the two equivalent forms

$$\boldsymbol{T}^{(1)}(\vec{X},t) = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t}(\vec{X},t) \,, \qquad \boldsymbol{t}(\vec{X},t) = J^{-1}\boldsymbol{F} \cdot \boldsymbol{T}^{(1)}(\vec{X},t) \,. \tag{3.21}$$

The surface-force vector $d\vec{g}$ in (3.20) acts upon the displaced point $\vec{x}$, whereas the surface-element vector $d\vec{A}$ is referred to the reference point $\vec{X}$. The first Piola-Kirchhoff stress tensor $\boldsymbol{T}^{(1)}$ is therefore a two-point tensor. This can also be observed from the componental form of (3.21):

$$T^{(1)}_{Kl}(\vec{X},t) = JX_{K,k}t_{kl}(\vec{X},t) \,, \qquad t_{kl}(\vec{X},t) = J^{-1}x_{k,K}T^{(1)}_{Kl}(\vec{X},t) \,. \tag{3.22}$$

The constitutive equations for a simple materials (see equation (5.37) in Chapter 5) are expressed most conveniently in terms of another measure of stress, known as the *second Piola-Kirchhoff stress tensor*. This quantity, denoted by $\boldsymbol{T}^{(2)}$, gives, instead of the actual surface force $d\vec{g}$ acting on the deformed area element $d\vec{a}$, a force $d\vec{G}$ related to $d\vec{g}$ in the same way as the referential differential $d\vec{X}$ is related to the spatial differential $d\vec{x}$. That is

$$d\vec{G} = \boldsymbol{F}^{-1} \cdot d\vec{g} \,, \tag{3.23}$$

in the same manner as $d\vec{X} = \boldsymbol{F}^{-1} \cdot d\vec{x}$. Defining $\boldsymbol{T}^{(2)}$ by

$$d\vec{G} =: d\vec{A} \cdot \boldsymbol{T}^{(2)} \,, \tag{3.24}$$

we find the first and second Piola-Kirchhoff stresses are related by

$$\boldsymbol{T}^{(2)} = \boldsymbol{T}^{(1)} \cdot \boldsymbol{F}^{-T} \,, \qquad \boldsymbol{T}^{(1)} = \boldsymbol{T}^{(2)} \cdot \boldsymbol{F}^{T} \,. \tag{3.25}$$

Comparing this result with (3.21), we obtain the corresponding relationship between the second Piola-Kirchhoff stress tensor and the Lagrangian Cauchy stress tensor:

$$\boldsymbol{T}^{(2)}(\vec{X},t) = J\boldsymbol{F}^{-1} \cdot \boldsymbol{t}(\vec{X},t) \cdot \boldsymbol{F}^{-T} \,, \qquad \boldsymbol{t}(\vec{X},t) = J^{-1}\boldsymbol{F} \cdot \boldsymbol{T}^{(2)}(\vec{X},t) \cdot \boldsymbol{F}^{T} \,. \tag{3.26}$$

Since the transformed surface force $d\vec{G}$ may be considered to act at the referential position $\vec{X}$ rather than at the spatial position $\vec{x}$, the second Piola-Kirchhoff stress tensor is an ordinary (a one-point) rather than a two-point tensor. This can also be seen from the componental form of (3.25):

$$T^{(2)}_{KL}(\vec{X},t) = JX_{K,k}X_{L,l}t_{kl}(\vec{X},t) \,, \qquad t_{kl}(\vec{X},t) = J^{-1}x_{k,K}x_{l,L}T^{(2)}_{KL}(\vec{X},t) \,. \tag{3.27}$$

The foregoing expressions may be used as a source for the linearized theory in which the displacement gradient $\boldsymbol{H}$ is much smaller when compared to unity, hence justifying linearization. To this end, we carry the linearized forms $(1.106)_{1,2}$ into $(3.21)_1$ and $(3.26)_1$ and obtain

$$\begin{aligned} \boldsymbol{T}^{(1)} &= (1 + \operatorname{tr}\boldsymbol{H})\,\boldsymbol{t} - \boldsymbol{H}^T \cdot \boldsymbol{t} + O(\|\boldsymbol{H}\|^2) \,, \\ \boldsymbol{T}^{(2)} &= (1 + \operatorname{tr}\boldsymbol{H})\,\boldsymbol{t} - \boldsymbol{H}^T \cdot \boldsymbol{t} - \boldsymbol{t} \cdot \boldsymbol{H} + O(\|\boldsymbol{H}\|^2) \,. \end{aligned} \tag{3.28}$$

The last equation demonstrates that the symmetry of tensor $\boldsymbol{T}^{(2)}$ has not been violated by linearization process. Conversely,

$$\begin{aligned} \boldsymbol{t} &= (1 - \operatorname{tr}\boldsymbol{H})\boldsymbol{T}^{(1)} + \boldsymbol{H}^T \cdot \boldsymbol{T}^{(1)} + O(\|\boldsymbol{H}\|^2) \\ &= (1 - \operatorname{tr}\boldsymbol{H})\boldsymbol{T}^{(2)} + \boldsymbol{H}^T \cdot \boldsymbol{T}^{(2)} + \boldsymbol{T}^{(2)} \cdot \boldsymbol{H} + O(\|\boldsymbol{H}\|^2)\ . \end{aligned} \tag{3.29}$$

Supposing, in addition, that stresses are small compared to unity (the *infinitesimal deformation and stress theory*), then

$$\boldsymbol{T}^{(1)} \cong \boldsymbol{T}^{(2)} \cong \boldsymbol{t}\ , \tag{3.30}$$

showing that, when considering infinitesimal deformation and stress, a distinction between the Cauchy and the Piola-Kirchhoff stresses is not necessary.